

1) Family of smooth curves.

Def $C \rightarrow S$ proper smooth.

$\forall$ geom fiber is a genus g curve.

Prop $\pi: C \rightarrow S$ family of smooth curve of genus $g \geq 2$.

i) $\pi_* \mathcal{O}_C = \mathcal{O}_S$

ii) $\pi_* \Omega_{C/S}^{\otimes k}$ vector bundle of rk $\begin{cases} g & k=1 \\ (2k-1)(g-1) & k>1. \end{cases}$

commutes w/ base change.

iii) $R^i \pi_* \Omega_{C/S}^{\otimes k} = \begin{cases} \mathcal{O}_S & k=1 \\ 0 & \text{o.w.} \end{cases}$

iv) $k \geq 3$ $\Omega_{C/S}^{\otimes k}$ is π -very ample.

pf). i, ii, iii) : Grauert + Riemann-Roch.

iv) $|3K_X|$ very ample for curves.

+ openness of ampleness.

2) Nodes. C/k curve.

Def $p \in C(k)$ is a node if

① if $k=\bar{k}$, $\hat{\mathcal{O}}_{C,p} \simeq k[x,y]/(x^2, y^2)$

② k non alg. closed. $\exists p' \in C(k)$ p' : node

Question?

Rmk maybe one should impose condition on other points / $k \dots ?$

3) Characterization of Nodes.

- Singular locus.

$M : R\text{-mod.}$

$$F_1 \xrightarrow{A} F_0 \rightarrow M \rightarrow 0$$

rank n .
↓
finite presentation.

$\text{Fit}_k(M) =$ ideal gen by $(n-k) \times (n-k)$ -minors of A .

(does not depend on the choice of $F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$).

M : loc free of rk r $\iff$ $\text{Fit}_{r-1}(M) = 0$, $\text{Fit}_r(M) = R$.

- C curve $\Rightarrow$ $\text{Sing}(C)$ defined by $\text{Fit}_1(\Omega_C)$
(scheme theoretically).

(locus where Ω_C fails to be locally free of rk 1).

Characterization of Nodes. continued

C : pure 1 dim curve / k . $p \in C$ closed pt.

let $p \in C$ be a node. Then.

- 1) C : l.c.i at p . geometric fibre is
↑ discrete & reduced.
- 2) ~~$k \rightarrow k(p)$~~ $\text{Sing}(C)$ is unramified / k at p .

- 3) $k \rightarrow k(p)$ is separable & $\hat{\mathcal{O}}_{C,p} \simeq k(p)[[x,y]]/(g)$.

g : non-deg quadratic form.

- 4) $\exists k \rightarrow k'$ sep. $\hat{\mathcal{O}}_{C,p} \otimes_k k' \simeq k[[x,y]]/(xy)$.

pt 1) Being l.c.i stable under base change & completion.
 $k[[x,y]]/(xy)$ ^{complete intersection} ~~is~~ nly.

2). Fitting ideal commutes w/ localization, completion.

$$\text{Sing}(\hat{\mathcal{O}}_{C_{\bar{k}}, \bar{p}}) = V(xy) \subset \text{+}$$

and this is $\text{Spec } \bar{k}$. $\therefore \text{Sing}(C) \rightarrow k$ unram.

3). Separability of $k \rightarrow k(p)$ comes from unram.

Structure thm for complete local nlys.

$$\hat{\mathcal{O}}_{C,p} \simeq k(p)[[x_1, \dots, x_n]]/(f_1, \dots, f_r).$$

unramified $\Rightarrow$ $(n-1) \times (n-1)$ minors of $\left(\frac{\partial f_j}{\partial x_i}\right)$
 generates $\underline{m} = (x_1, \dots, x_n) \in \hat{O}_{C,p}$.

If $\frac{\partial f_j}{\partial x_i}$ unit. $\Rightarrow$ change coordinates $\swarrow$ gen $\underline{m}/\underline{m}^2$.
 $x_1, \dots, \hat{x}_i, \dots, x_n, f_j$.

$$\hat{O}_{C,p} \simeq K(p) \llbracket x_1, \dots, \hat{x}_i, \dots, x_n, f_j \rrbracket / (f_1, \dots, f_{n-1})$$

$\Rightarrow$ # of gen drops by 1.

$\therefore$ assume $\frac{\partial f_j}{\partial x_i} \in \underline{m}$ for all i, j . $\Rightarrow \text{Fit}_1(\omega_C) \subseteq \underline{m}^{n-1}$.

$$\Rightarrow \underline{m} = \underline{m}^2. \quad \therefore \hat{O}_{C,p} = K(p) \llbracket x, y \rrbracket / (g).$$

$$g: \text{non-0} \quad g \in \underline{m}^2.$$

$$g = g_2 + g_3 + \dots \quad g_2 = ax^2 + by + cy^2.$$

$$\text{rearrange coordinates s.t. } \hat{O}_{C,p} \simeq K(p) \llbracket x, y \rrbracket / (g_2).$$

Idea enough to show $x_0, y_0 \in \underline{m}$. (generating $\underline{m}$)

$$\text{s.t. } g_2(x_0, y_0) = 0.$$

Inductively build $x_i \in \underline{m}^{i+1}$ s.t.

$$g_2(x_0 + \dots + x_N, y_0 + \dots + y_N) \in \underline{m}^{N+3}.$$

$$\text{get } \bar{x} = \sum x_i \quad \& \quad \bar{y} = \sum y_i \quad \text{mod } \underline{m}^{N+3}.$$

$$g(x' + x_N, y' + y_N) = g(x', y') + (2ax_0 + by_0)x_N + (bx_0 + 2cy_0)y_N \quad \textcircled{4}$$

4) $k \subset k' \rightarrow k'$ s.t. g_2 splits into linear factors.

$$\hat{\mathcal{O}}_{C,p} \otimes k' \simeq k' \llbracket x, y \rrbracket / (xy).$$

□

4. Local structure. C/k node at $p \in C$.

$\exists k \rightarrow k'$ Art. sep.

$$\begin{array}{ccc} & (u, u) & \\ \swarrow \text{Art} & & \searrow \text{Art} \\ (C, p) & & (\text{Spec } k'[xy]/(xy), o) \end{array}$$

$\therefore$ Artin approximation.

5. Genus formula. C connected, nodal, proj / $k = \bar{k}$.

$p_1, \dots, p_\delta \in C$ nodes.

$C = \bigcup_{i=1}^r C_i$ in comp.

$g_i = g(\tilde{C}_i)$ $\tilde{C}_i \rightarrow C_i$ normalization.

$$\Rightarrow g = \sum g_i + \delta - r + 1.$$

pf) $\tilde{C} = \coprod \tilde{C}_i \xrightarrow{\pi} C.$

$$0 \rightarrow \mathcal{O}_C \rightarrow \pi_* \mathcal{O}_{\tilde{C}} \rightarrow \bigoplus k(p_i) \rightarrow 0.$$

Take long exact sequence!

6)

6. Dualizing Sheaf.

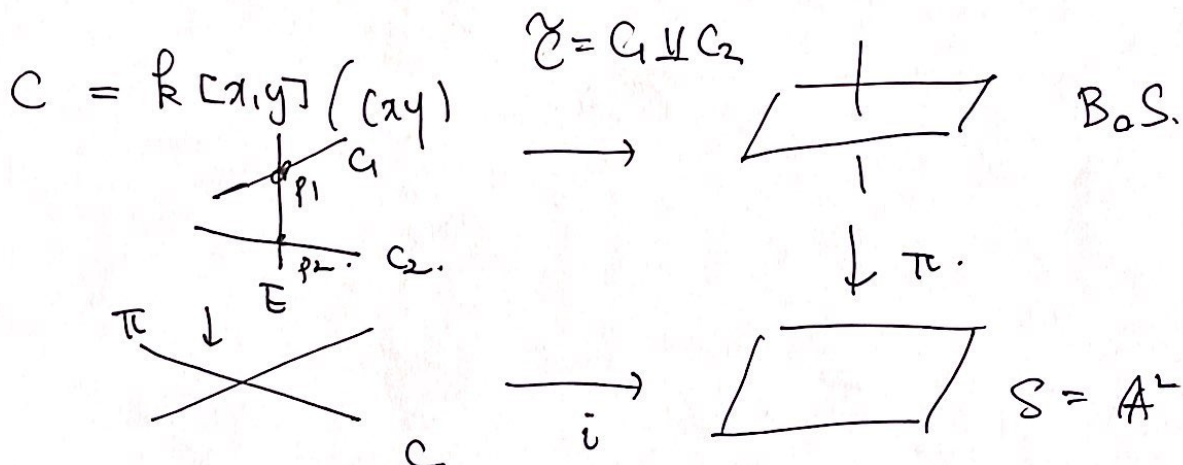
$C: \mathcal{O}_C \rightarrow \omega_C$ is a line bundle & can compute via adjunction

(Hartshorne III.7)

(compute étale locally.)

$$\therefore f: C' \xrightarrow{\text{ét}} C.$$

$$\omega_{C'} = f^* \omega_C.$$



$$\pi^* \omega_C = \pi^* (\omega_{\mathbb{A}^2} + E) |_{\tilde{C}}$$

$$= (\omega_{B_0 S} - E) + \tilde{C} + 2E |_{\tilde{C}} = \omega_{\tilde{C}}(E).$$

$$= \omega_{\tilde{C}}(p_1 + p_2)$$

$$\alpha \mapsto \omega_C \rightarrow \pi_* \omega_{\tilde{C}}(p_1 + p_2) \rightarrow K \rightarrow 0$$

ω_C subsheaf of $\pi_* \omega_{\tilde{C}}(p_1 + p_2)$ whose residues at p_1 & p_2 agree.

Remark This globalizes since normalization commutes w/
~~stable~~ base change. $\square$

$\nu: \tilde{C} \rightarrow C$ norm. $p_1, \dots, p_s$ nodes.

$g_i, n_i \rightarrow p_i$ preimage

$$\omega_C \subset \nu_* \omega_{\tilde{C}} (\Sigma g_i + n_i)$$

where $\alpha \in \omega_C \iff \text{res}_{g_i}(\alpha) = \text{res}_{n_i}(\alpha) \forall i.$

7. Family of nodal curves

$$C \rightarrow S$$

Local structure: $C \rightarrow S$ fppf, $\forall$ geometric fibre B a curve.
 $p \in C_s$ node.

$$\begin{array}{ccccc} (C, p) & \xleftarrow{\text{ét}} & (U, u) & \xrightarrow{\text{ét}} & \left(\text{Spec } \frac{A[x,y]}{(xy-f)}, (s', o) \right) \\ \downarrow & & \downarrow & & \swarrow \\ (S, s) & \xleftarrow{\text{ét}} & (\text{Spec } A, s') & & \end{array} \quad \& \quad f \in A \text{ such that } f(s') = 0.$$

1) S finite type / $\mathbb{Z}$, affine. (limit).

$\left(\begin{array}{l} \text{We need. "fppf + } \forall \text{ fibre has pure dim 1"} \\ \text{descends under limits} \end{array} \right)$

(7)

2). $K(S) \rightarrow k'$ sep base change. $\hat{\mathcal{O}}_{C,p} \otimes_{K(S)} k' \simeq \frac{k'[x,y]}{(xy)}$.

assume $\hat{\mathcal{O}}_{C,p} = \frac{K(S)[x,y]}{(xy)}$ after $S' \rightarrow S$

étale base change s.t. $K(S') \simeq k'$.

3) $\hat{\mathcal{O}}_{C,p} \simeq \hat{\mathcal{O}}_{S,s} [x,y] / (xy - \hat{f}) \quad \hat{f} \in \underline{m}_s \subset \hat{\mathcal{O}}_{S,s}$.

Goal construct x_n, y_n, f_n inductively s.t.

$$(\mathcal{O}_{S,s} / \underline{m}_s^{n+1}) [X,Y] / (XY - f_n) \simeq \mathcal{O}_{C,p} / \underline{m}_s^{n+1} \mathcal{O}_{C,p}.$$

Take $x_n \rightarrow \hat{x}, y_n \rightarrow \hat{y}, f_n \rightarrow \hat{f}$.

4). Remark The completion of $\mathcal{O}_{C,p}$ w.r.t. $\underline{m}_s$ -adic topy.

4).
$$\left(\text{Spec } \hat{\mathcal{O}}_{S,s} \times_S C, (s,p) \right) \xrightarrow{\text{ét}} (U,u) \xrightarrow{\text{ét}} \left(\text{Spec } \frac{\hat{\mathcal{O}}_{S,s}[x,y]}{(xy - \hat{f})}, (s,p) \right)$$

by Artin-Approximation.

Now, we need to make this as an S -alg.

$S = \text{Spec } A$. A : excellent ($A \rightarrow \hat{A}$ geometrically regular).

Néron-Popescu: $\hat{A} = \varinjlim B_\lambda$ B_λ : smooth A -alg.

(8)

$$\lambda \gg 0. \quad \hat{f} = f_\lambda \in B_\lambda, \quad x, y \in T(U, \mathcal{O}_U) \\ \stackrel{\cap}{T}(U_\lambda, \mathcal{O}_{U_\lambda}).$$

$$U_\lambda = U \times_S S_\lambda \text{ "Spec } B_\lambda.$$

$$\begin{array}{ccc} & \textcircled{1} & \textcircled{2} \\ & U_\lambda & \\ \swarrow & & \searrow \\ \mathcal{O}_{U_\lambda} S_\lambda =: \mathcal{C}_\lambda & & \text{Spec } B_\lambda[x, y] / (xy - f_\lambda) \\ & \searrow & \swarrow \\ & \text{Spec } B_\lambda & \end{array}$$

(base change by $B_\lambda \rightarrow \hat{\mathcal{O}}_{S, s}$ gives the original one).

$\textcircled{1}, \textcircled{2}$ are étale for $\lambda \gg 0$.

$$\begin{array}{ccccc} (\mathcal{C}, p) & \xleftarrow{*} & (U_\lambda, u_\lambda) & \xrightarrow{\text{ét}} & (\text{Spec } B_\lambda[x, y] / (xy - f_\lambda)) \\ \downarrow & & \downarrow & & \swarrow \\ (S, s) & \xleftarrow{*} & (\text{Spec } B_\lambda, s_\lambda) & & \end{array}$$

* are smooth morphisms. $\Rightarrow$ get a section étale locally.

$$\begin{array}{ccc} (S, s) & \xleftarrow{\quad} & \text{Spec } B_\lambda \\ & \nwarrow \text{ét} & \uparrow \\ & & S' \end{array} \quad \& \text{ base change by } S' \text{ v.}$$

□

⑨